

CERTAIN PROPERTIES OF A GYROCOMPASS WITH A SEVENTEEN-HOUR PERIOD

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It is well known that existing gyrocompasses possess two possible positions of equilibrium. If the velocity deviations and the deviations of damping are neglected, in the first position the momentum vector of the gyroscopic system is directed north, in the second position it is directed south. The first position is the principal working position of the gyrocompass and it is always stable. In existing gyrocompasses the second position is unstable.

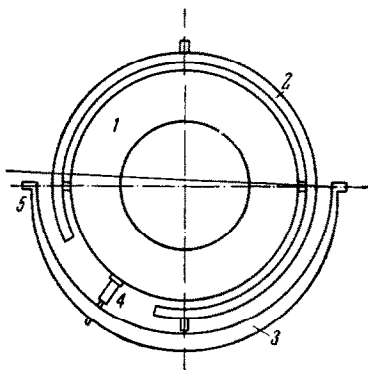


Fig. 1.

We shall demonstrate that a gyrocompass, with parameters causing a considerable increase of its free vibration period, may have in general four positions of equilibrium. A gyrocompass working on the Earth's surface will display this phenomenon when its period of free vibrations about its principal (northern) position of equilibrium is larger than 17 hours.

We shall consider a gyrocompass with one rotor, resembling the first Sperry gyrocompass (study of other types of gyrocompasses will disclose similar properties). In Fig. 1 the numeral 1 indicates the gyroscope's casing which contains the spinning rotor, the numeral 2 indicates the outer gimbal frame, the numeral 3 the correcting pendulum.

Moments arising from the pendulum swinging from its vertical position are transmitted to the gyroscope through the eccentrically located pin 4.

The pendulum is on a separate suspension and has a tracking system. The tracking system combines the axis of the pendulum's suspension 5 with the axis of the casing's suspension in the outer frame.

The orientation of the axis of the gyroscope is arbitrary. We shall introduce two coordinate systems with common origin at the rotor's center of gravity O .

The system $x_0 y_0 z_0$ is a geographically oriented trihedron on the Earth's surface (Fig. 2).

The x_0 -axis is horizontal and directed north, the y_0 -axis is vertical and directed upwards. If the gyrocompass is on a stationary base, then the angular velocity of the trihedron $x_0 y_0 z_0$ with respect to an inertial system would equal the angular velocity of the Earth's daily rotation ω_* . Resolving the Earth's angular velocity vector ω_* into the local horizontal (ω_*') and vertical (ω_*'') components, we can obtain the z_0 - and the y_0 -components.

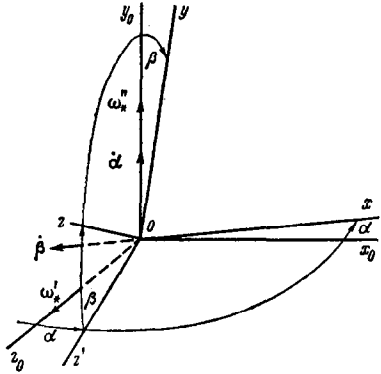


Fig. 2.

The xyz -system is fixed in the casing; the x -axis coincides with the rotation axis of the casing, the z -axis coincides with the angular momentum vector of the gyroscope. We shall assume that the axis of the outer gimbal frame is stabilized in the direction of the local vertical, that is, in the direction of y_0 . With this assumption the x -axis will always be in the $x_0 y_0$ -plane. The orientation of the xyz -system with respect to the $x_0 y_0 z_0$ -system is determined through angles α and β . The angle α is the rotation angle of the outer gimbal frame about its suspension axis, the angle β is the rotation angle of the casing with respect to the outer frame. The angles α and β are measured from this position of the gyroscope when the z -axis coincides with z_0 -axis. The positive direction for the angles α and β is indicated in Fig. 2.

The absolute angular velocity (velocity with respect to an inertial system) of the trihedron xyz is

$$\omega = \omega_*' + \omega_*'' + \dot{\alpha} + \dot{\beta}$$

The meaning of the vectors $\dot{\alpha}$ and $\dot{\beta}$ just introduced here is clear from Fig. 2. The x - and y -components of the vector ω_* are

$$\omega_x = -\omega_*' \sin \alpha - \dot{\beta}, \quad \omega_y = \omega_*'' \cos \beta - \omega_*' \cos \alpha \sin \beta + \dot{\alpha} \cos \beta$$

We assume that the gyroscope is in a steady regime

$$M_{z\Sigma} = 0, \quad \Gamma = \text{const} \quad (1)$$

Here $M_{z\Sigma}$ is the z -component of the principal moment of the external forces acting on the gyroscope, Γ is the angular momentum of the gyroscope.

Assuming that the tracking system which connects the axis of the correcting pendulum with the casing's axis is mounted without friction, we find the x -, y -, and z -components of the pendulum's moment. We have

$$M_x = pl \sin \beta, \quad M_y = pl \frac{l_2}{l_1} \sin \beta, \quad M_z = 0 \quad (2)$$

Here p is the weight of the pendulum, l is the distance from the suspension axis to the center of gravity of the pendulum. The meaning of l_1 and l_2 is clear from Fig. 3.

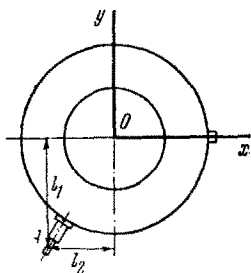


Fig. 3.

The moments of friction in the suspension axes of the gyroscope and of the pendulum will be neglected. In the region corresponding to these values of β which we shall consider here, the reaction moment \mathbf{M}_r acting on the outer gimbal frame can reach considerable values. In our gyrocompass this moment equals zero for all values of β with the exception of $\beta = \pm \pi/2$.

Let us assume that $\mathbf{M}_r \neq 0$. With frictionless suspension axes of the gyroscope, the vector moment \mathbf{M}_r should be perpendicular to the plane of the outer frame. This moment has a finite z -component for all values of β , with the exception of $\beta = \pm \pi/2$. If we take into consideration (2), we can see that this contradicts the condition (1). Consequently, \mathbf{M}_r must equal zero for all values of β , with the exception of $\beta = \pm \pi/2$.

The x - and y -components of the equations of motion of the gyroscope are

$$\Gamma \omega_*'' \cos \beta - \Gamma \omega_*' \cos \alpha \sin \beta + \Gamma \dot{\alpha} \cos \beta = k \sin \beta \quad (3)$$

$$\Gamma \omega_*' \sin \alpha + \Gamma \dot{\beta} = -h \sin \beta$$

Here $k = pl$, $h = pll_2/l_1$. We shall determine all possible positions of equilibrium of the gyroscope. Setting $\dot{\alpha} = \dot{\beta} = 0$, we obtain

$$(\Gamma \omega_*' \cos \alpha_0 + k) \sin \beta_0 - \Gamma \omega_*'' \cos \beta_0 = 0, \quad \Gamma \omega_*' \sin \alpha_0 + h \sin \beta_0 = 0 \quad (4)$$

Here α_0 and β_0 are the values of α and β in a position of equilibrium.

We shall consider first the positions of equilibrium of the gyrocompass which occur when the angular momentum vector of the gyroscope is directed approximately north or south. Near the northern position of equilibrium the angles α and β are small.

We obtain

$$(\Gamma\omega_*' + k)\beta_0 - \Gamma\omega_*'' = 0, \quad \Gamma\omega_*'\alpha_0 + h\beta_0 = 0$$

Hence

$$\alpha_0 = -\frac{h}{\Gamma\omega_*' + k} \frac{\omega_*''}{\omega_*'}, \quad \beta_0 = \frac{\Gamma\omega_*''}{\Gamma\omega_*' + k} \quad (5)$$

The relation (5) gives the value of the normal angle of elevation of the gyroscope's axis with respect to the plane of the horizon (the level plane) and the value of the damping deviation. We note that the obtained values of α_0 and β_0 will not remain small if the parameters of the instrument are allowed to vary within large bounds. The case when the parameter k is allowed to decrease considerably will be studied presently. The linearization of the equations (4) may not be justified in this case.

While considering small values of k we shall also assume that the inequality $\omega_*'' \ll \omega_*'$ is satisfied. This means that the gyrocompass works in low latitudes. With the above inequality satisfied, the linearization of the equations (4) near the northern position of equilibrium is justified for all values of k .

Linearizing similarly the equations of motion (3) we find

$$\Gamma\ddot{\alpha} - (\Gamma\omega_*' + k)\beta + \Gamma\omega_*'' = 0, \quad \Gamma\dot{\beta} + \Gamma\omega_*'\alpha + h\beta = 0$$

We shall turn our attention now to the equations which control perturbations

$$\Gamma\Delta\dot{\alpha} - (\Gamma\omega_*' + k)\Delta\beta = 0, \quad \Gamma\Delta\dot{\beta} + \Gamma\omega_*'\Delta\alpha + h\Delta\beta = 0 \quad (6)$$

Here $\Delta\alpha = \alpha - \alpha_0$ and $\Delta\beta = \beta - \beta_0$ are the perturbations of α and β . The characteristic equation of the system (6)

$$p^2 + \frac{h}{\Gamma}p + \frac{(\Gamma\omega_*' + k)\omega_*''}{\Gamma} = 0 \quad (7)$$

shows that the northern position of equilibrium of the gyrocompass axis is asymptotically stable for all positive values of h , k and Γ . In order to study the southern position of equilibrium we make the substitution $\alpha' = \pi - \alpha$. Assuming that the angles α' and β are small, we obtain

$$\Gamma \dot{\alpha}' + (k - \Gamma \omega_*') \beta - \Gamma \omega_*'' = 0, \quad \Gamma \dot{\beta} + h\beta + \Gamma \omega_*' \alpha' = 0$$

In the position of equilibrium

$$\alpha_0' = - \frac{h}{k - \Gamma \omega_*'} \frac{\omega_*''}{\omega_*'}, \quad \beta_0 = \frac{\Gamma \omega_*''}{k - \Gamma \omega_*'} \tag{8}$$

The perturbed equations are

$$\Gamma \Delta \dot{\alpha}' + (k - \Gamma \omega_*') \Delta \beta = 0, \quad \Gamma \Delta \dot{\beta} + h \Delta \beta + \Gamma \omega_*' \Delta \alpha' = 0$$

The characteristic equation

$$p^2 + \frac{h}{\Gamma} p - \frac{\omega_*' (k - \Gamma \omega_*')}{\Gamma} = 0$$

shows that the investigated position of equilibrium is unstable only when $k \geq \Gamma \omega_*'$. When $k < \Gamma \omega_*'$, then the position of equilibrium (8) becomes stable and the kinetic axis of the gyrocompass is capable of being stabilized in southern position.

Using Equation (7) and assuming that the coefficient h which determines the damping of vibrations is small, we can calculate the period of free vibrations of the instrument near its principal (northern) position of equilibrium from the formula

$$T = \frac{2\pi}{\sqrt{(\Gamma \omega_*' + k) \omega_*' / \Gamma}}$$

Substituting in the above formula $k = \Gamma \omega_*'$, we find the free vibrations of the instrument in this unstable southern position of equilibrium which is on the very border of the region of instability. We obtain

$$T' = \sqrt{2} \frac{\pi}{\omega_*'}$$

Consider for example the gyrocompass placed at the Earth's equator. Substituting $\omega_*' = \omega_* = 7.27 \times 10^{-5} \text{ sec}^{-1}$ we obtain

$$T_1 = 16 \text{ hrs } 58.3 \text{ min} \approx 17 \text{ hrs}$$

Let us consider Equations (4). Eliminating from them the angle α we obtain

$$(u = \tan \beta) \tag{9}$$

This equation can have in general four real roots, therefore we can expect in a general case that the axis of a gyrocompass will have four positions of equilibrium in space. Consider again a gyrocompass placed at the equator. In this case $\omega''_* = 0$, $\omega'_* = \omega_*$, and the equation (9) will take the form

$$(k^2 + h^2 - \Gamma^2\omega_*^2) u^4 + (k^2 - \Gamma^2\omega_*^2) u^2 = 0 \quad (u = \tan \beta) \quad (10)$$

Solving Equation (10) we find

$$u_1 = 0, \quad u_2 = 0$$

$$u_3 = \sqrt{\frac{k^2 - \Gamma^2\omega_*^2}{k^2 + h^2 - \Gamma^2\omega_*^2}}, \quad u_4 = -u_3 = -\sqrt{\frac{k^2 - \Gamma^2\omega_*^2}{k^2 + h^2 - \Gamma^2\omega_*^2}}$$

Utilizing the above expression, we can obtain the following four solutions of the system (4):

$$\alpha_{01} = 0, \quad \alpha_{02} = \pi, \quad \alpha_{03} = \cos^{-1} \left(-\frac{k}{\Gamma\omega_*} \right), \quad \alpha_{04} = -\cos^{-1} \left(-\frac{k}{\Gamma\omega_*} \right)$$

$$\beta_{01} = 0, \quad \beta_{02} = 0, \quad \beta_{03} = \sin^{-1} \left(-\frac{\sqrt{\Gamma^2\omega_*^2 - k^2}}{h} \right), \quad \beta_{04} = \sin^{-1} \frac{\sqrt{\Gamma^2\omega_*^2 - k^2}}{h}$$

The positions of equilibrium corresponding to the first two solutions have already been discussed. The position $(\alpha_{01}, \beta_{01})$ is the principal

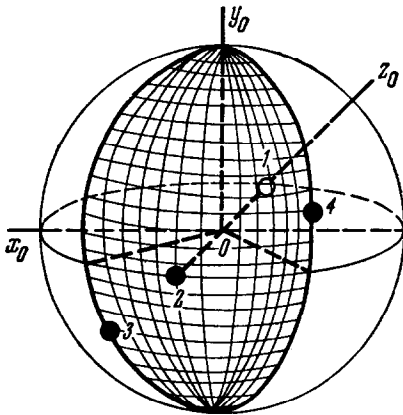


Fig. 4.

(northern) position of equilibrium of a gyrocompass and it is always stable. The position of equilibrium $(\alpha_{02}, \beta_{02})$ corresponds to the southern position of equilibrium of the gyrocompass axis, and it is unstable when $k > \Gamma\omega_*$ and stable when $k < \Gamma\omega_*$. Two new positions of equilibrium corresponding to the solutions $(\alpha_{03}, \beta_{03})$ and $(\alpha_{04}, \beta_{04})$ are located on the phase sphere (α, β) ; when looking at it from the side of the principal position of equilibrium, that is, from the side of the z_0 -axis (Fig. 4), they are on its rear half. The positions of equilibrium $(\alpha_{03}, \beta_{03})$ and $(\alpha_{04}, \beta_{04})$ are located symmetrically with respect to the z_0 -axis. These positions of equilibrium arise only

when $k < \Gamma\omega'_*$; that is, only when the position of equilibrium $(\alpha_{02}, \beta_{02})$ becomes stable.

Let us investigate the stability of the position of equilibrium $(\alpha_{03}, \beta_{03})$ and $(\alpha_{04}, \beta_{04})$. Expanding the terms in Equations (3) in series about some position of equilibrium, say (α_0, β_0) , and neglecting all terms of third order and higher, we obtain

$$\begin{aligned} \Delta \dot{\alpha} &= \omega_* \left[\left(\cos \alpha_0 + \frac{k}{\Gamma \omega_*} \right) \tan \beta_0 - \sin \alpha_0 \tan \beta_0 \Delta \alpha + \sec^2 \beta_0 \left(\cos \alpha_0 + \frac{k}{\Gamma \omega_*} \right) \Delta \beta \right] \\ \Delta \dot{\beta} &= -\omega_* \left(\sin \alpha_0 + \cos \alpha_0 \Delta \alpha + \frac{h}{\Gamma \omega_*} \sin \beta_0 + \frac{h}{\Gamma \omega_*} \cos \beta_0 \Delta \beta \right) \end{aligned}$$

or, using the condition (4)

$$\begin{aligned} \Delta \dot{\alpha} + \omega_* \sin \alpha_0 \tan \beta_0 \Delta \alpha - \omega_* \sec^2 \beta_0 \left(\cos \alpha_0 + \frac{k}{\Gamma \omega_*} \right) \Delta \beta &= 0 \\ \omega_* \cos \alpha_0 \Delta \alpha + \Delta \dot{\beta} + \frac{h}{\Gamma} \cos \beta_0 \Delta \beta &= 0 \end{aligned}$$

Here $\Delta \alpha = \alpha - \alpha_0$, $\Delta \beta = \beta - \beta_0$ are small deviations of α and β from their equilibrium values. The characteristic equation of the perturbed system of equations is

$$\begin{aligned} p^2 + \left(\sin \alpha_0 \tan \beta_0 + \frac{h}{\Gamma \omega_*} \cos \beta_0 \right) \omega_* p + \\ + \left[\frac{h}{\Gamma \omega_*} \sin \alpha_0 \sin \beta_0 + \sec^2 \beta_0 \cos \alpha_0 \left(\cos \alpha_0 + \frac{k}{\Gamma \omega_*} \right) \right] \omega_*^2 = 0 \end{aligned} \quad (11)$$

Substituting in (11) the values of trigonometric functions corresponding to the position of equilibrium $(\alpha_{03}, \beta_{03})$ we obtain

$$p^2 + \frac{2k^2 + h^2 - 2\Gamma^2 \omega_*^2}{\sqrt{h^2 + k^2 - \Gamma^2 \omega_*^2}} \omega_* p - (\Gamma^2 \omega_*^2 - k^2) = 0 \quad (12)$$

The position of equilibrium $(\alpha_{03}, \beta_{03})$ exists only when $k < \Gamma \omega_*$, therefore if it exists it must be unstable. Since the free term in the characteristic equation (12) is negative, we have a saddle point at the position of equilibrium $(\alpha_{03}, \beta_{03})$. Exactly the same results are obtained for the position of equilibrium $(\alpha_{04}, \beta_{04})$.

Thus, when the period of free vibrations near the principal (northern) position of equilibrium reaches seventeen hours, then a gyrocompass of the investigated type and located at the equator can have four different positions of equilibrium. Two of them occur at the northern or southern orientations of the kinetic axis of a gyrocompass, and they are stable. The remaining two positions of equilibrium are located on the southern

half of the space hemisphere and are special points of the saddle-type.

Figure 5 shows an example of mapping of the phase sphere on the (α, β) phase plane for a gyroscope of the investigated type. The inequality $k < \Gamma\omega_*$, necessary for the existence of the positions of equilibrium

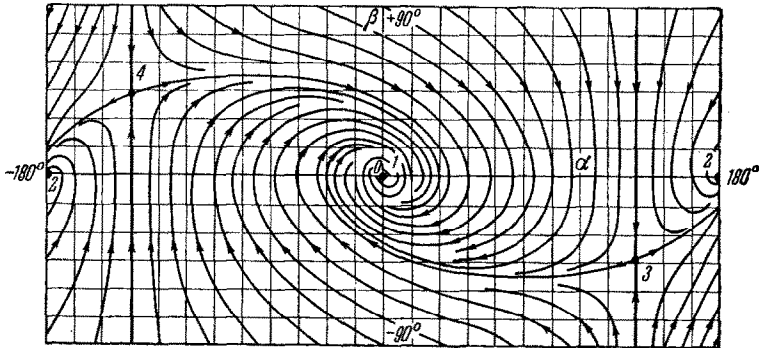


Fig. 5.

$(\alpha_{03}, \beta_{03})$ and $(\alpha_{04}, \beta_{04})$ is assumed here to be satisfied. The mapping shows four positions of equilibrium of the instrument. The points $(+180.0)$ and (-180.0) represent the same position of equilibrium $(\alpha_{02}, \beta_{02})$ (the southern one), and here it is stable. Thick lines denote separatrices, which separate the region of stability of the principal position of equilibrium $(\alpha_{01}, \beta_{01})$ from the region of stability $(\alpha_{02}, \beta_{02})$. Reverting to the equations of motion (3), we can easily convince ourselves that on the sphere α, β the separatrices are arcs of the large circles given by the parametric equations

$$\alpha = \cos^{-1}(-k/\Gamma\omega_*) \quad \text{and} \quad \alpha = -\cos^{-1}(-k/\Gamma\omega_*)$$

In Fig. 4 the separatrices are thickly drawn curves, and the region of stability of the position of equilibrium $(\alpha_{02}, \beta_{02})$ is shaded.

Translated by T.L.